Unusual $1/r$-dependent radiation intensity in any biaxial crystal

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Lighthill's prediction of an unusual $1/r$-dependent radiation intensity from a small monochromatic source is found to exist in any homogeneous and lossless biaxial crystal along the two optical ray axes, where r is the radial distance from the source. This is a consequence of the special shape of the wave-vector surface on which there is, around each singular point of self-intersection (or ''dimple''), a circular locus of points all having the same surface normal direction and thus sharing a common tangent plane. A heuristic derivation of the result is given and a simple optical experiment is proposed to detect such an unusual distance dependence.

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I. INTRODUCTION

It is well known that the intensity of radiation emitted from a light source in space decreases with distance according to the inverse square law as required by energy conservation and the idea of ray emission. However, for a source in an anisotropic medium, there is no violation of any physical law if the intensity within the medium decreases at a slower radial dependence only along a certain direction, and hence by no means within a finite solid angle. Lighthill, in a 1960 paper mainly addressed to calculation of the radiation field caused by a localized single-frequency source [1], appears to be the first to have predicted this peculiarity. He applied the method of stationary phase and showed that the unusual slower radial dependence is a direct consequence of zero Gaussian curvature at certain points on the wave-vector surface (WS) [2], which is the surface in wave-vector space satisfying the dispersion relation at a given frequency. In particular, he has shown that, in a medium whose WS has on it a curve where the surface normals are all parallel to one another, i.e., the points on the curve all share a common tangent plane, the radiation field along that normal direction decreases as $1/\sqrt{r}$, where r is the radial distance from the source, therefore implying a $1/r$-dependent radiation intensity. A real physical system that exhibits such an unusual characteristic was found by Wang and Bell [3] and by Giles [4] at a later time. It is a magnetoplasma with density and external magnetic field in certain ''parameter ponds'' of the high-frequency Clemmow-Mullaly-Allis (CMA) diagram [5], and the curve on the WS is a circle where the surface normals are all directed along the external magnetic field. The study was extended by Lai *et al.* [6] to a two-component magnetoplasma including low-frequency cases, where the ponds in the full CMA diagram were analytically classified into parameter domains for the occurrence of $1/r$-dependent radiation intensity, and the far field, the radiation flux, and other interesting characteristics were also studied.

Yet so far there has not been any experimental confirmation of the prediction. This is perhaps due to the difficulty of preparing a homogeneous magnetoplasma that is large enough for the study of the radial dependence of radiation at a distance from a source. We therefore address two issues in this article: (i) the existence of the $1/r$-dependent radiation intensity in another medium, namely, a biaxial crystal, and (ii) the possibility of a simple experimental detection with such a medium. We first describe, in the next few paragraphs, Lighthill's theory in a heuristic way. We then review the shape of the WS for a biaxial medium and point out the existence of the unusual radial dependence of the radiation intensity in any such medium. Finally, we estimate the strength of the radiation for certain biaxial crystals and describe a simple optical experiment to detect the phenomenon.

II. HEURISTIC DERIVATION OF THE UNUSUAL RADIAL DEPENDENCE

An extraneous monochromatic small source in a medium excites waves of the same frequency (ω, say) but of many different wave vectors ($\mathbf{k}$'s); each wave is of the form $\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$. All these excited waves superpose to give the total field at the position $\mathbf{r}$. For them to travel a long distance, they have to satisfy the dispersion relation. Therefore, only those waves whose $\mathbf{k}$'s (or their vector tips) lie on the WS contribute to the far field. This means that the superposition is in fact given by a surface integral over the WS. Mathematically, the far field, which can be obtained from Maxwell's equations through the method of Fourier transform, is of the form

$$\mathbf{E}(\mathbf{r})\propto\omega^2\int\int_{WS}d^2n\,\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{\mathbf{\Gamma}\cdot\mathbf{J}(\mathbf{k})}{\partial D/\partial n_g},\qquad(1)$$

where $\mathbf{n}\equiv c\mathbf{k}/\omega$ is the refractive index vector or the normalized wave vector, $\mathbf{J}(\mathbf{k})$ is the (spatial) Fourier transform of the extraneous source current density, D is the dispersion function of $\mathbf{k}$ and ω given by the determinant of the dispersion tensor $\Delta_{ij}\equiv\epsilon_{ij}+n_in_j-n^2\delta_{ij}$ with ϵ_{ij} being the dielectric tensor, $\mathbf{\Gamma}/D$ is the inverse of $\mathbf{\Delta}$, $\partial D/\partial n_g$ is the gradient of D with respect to $\mathbf{n}$ along the direction of the group ve-

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locity $\mathbf{V}_g$, and the integration is over that part of the WS satisfying $\mathbf{r}\cdot\mathbf{V}_g > 0$. (See [1], [4], [6], and [7].) Note that the exponential time factor has been omitted in the equation.

Waves can superpose constructively or destructively. This is related to the concept of group velocity. For waves with their $\mathbf{k}$'s in the neighborhood of a point on the WS, the group velocity is parallel to the surface normal at the point, meaning that these waves superpose to give an interference maximum along the normal direction. As a result, the field at a large distance $\mathbf{r}$ from the source comes predominantly from the waves of $\mathbf{k}$'s around the point (or points, if applicable) on the WS where the surface normal is parallel to and the group velocity is directed along $\mathbf{r}$. Only these waves most constructively superpose at $\mathbf{r}$. Note that the phase $\mathbf{k}\cdot\mathbf{r}$ assumes a stationary value at the point, which has been conveniently referred to as the point of stationary phase (PSP) with respect to the observation direction. The domain of the surface integral in Eq. (1) can therefore be reduced to the neighborhood of the PSP.

To examine the distance dependence of the far field, we consider any direction from the source and take it as the z axis. The phase of the waves at a point on the z axis becomes $k_z z$, where z, always positive by definition, is the distance from the source. Since the distance variable only appears in the phase factor [see Eq. (1)], how the far field varies with z really depends on how k_z changes on the WS (i.e., the shape of the WS) around the PSP. Take, for example, an extreme case where the WS has a finite flat portion around the PSP. Obviously, k_z is constant over the flat portion and the strength of the far field along the z axis is consequently z independent. This is simply a superposition of in-phase sinusoidal waves of the same wavelength along the same direction. Of course the extreme example is somewhat unrealistic. A usual case is a convex shape of the WS around the PSP. Suitable orthogonal surface parameters, u and v, say, can be chosen to expand k_z about the PSP, taken to be $(u,v) = (0,0)$. This subsequently simplifies the surface integral to a product of two similar line integrals around the PSP,

$$\int du \exp[i\kappa_u z u^2/2] \tag{2}$$

and another for v, where κ_u (κ_v) is the principal curvature of the u- (v-) parameter line on the surface at the PSP. (The u- and v-parameter lines are in fact the lines of curvature [8].) This is essentially the method of stationary phase in the two-dimensional case. While it is well known that each line integral gives a $1/\sqrt{z}$-dependent result for large z, it is worth noting that the integrand in Eq. (2) is oscillating with a fast-decreasing wavelength as u increases; the rapid oscillations nullify the result for larger u. Therefore, only the domain around $u=0$ contributes most to the integral. The effective width of this domain may be set equal to $\sqrt{2\pi/(|\kappa_u|z)}$, which is exactly the magnitude of the integration result in Eq. (2). Therefore, the effective area, being the product of the two effective widths, is $2\pi/(\sqrt{K}z)$, where $K \equiv |\kappa_u\kappa_v|$ is the Gaussian curvature. This leads to the usual $1/z$-dependent far field and $1/z^2$-dependent radiation intensity. We see that the farther away the observation point, the smaller the effective area around the PSP, or the fewer waves that contribute to the far field in a constructive way.

We now consider a case somewhat in between the previous two, a case that is our main concern in this article. This is the case where the WS has a depression surrounded by a circular "ridge" everywhere of the same "height." The circular line on the ridge is actually the locus of the PSP's all sharing a common tangent plane and having the same normal direction, say again the z direction. Here the circle may be considered as one of the v-parameter lines and the lines transverse to it as the u-parameter lines. Obviously, k_z changes only along the transverse direction and is therefore only u^2 dependent about the circle of the PSP's. We thus see that a whole circular band of wave vectors, instead of a neighborhood around a single point, contribute to the far field. The effective area is therefore equal to the finite circumference times the width $(|\kappa_u|z)^{-1/2}$, leading readily to a $1/\sqrt{z}$-dependent far field, and hence a $1/z$-dependent radiation intensity along the z axis.

III. APPLICATION TO A BIAXIAL CRYSTAL

Consider the WS of a biaxial crystal which, in terms of the coordinate system of the principal dielectric axes, the $X_1X_2X_3$ system, say, is given by the zero of the following dispersion function

$$D(\mathbf{n}) = (\epsilon_1 n_1^2 + \epsilon_2 n_2^2 + \epsilon_3 n_3^2)n^2 - \epsilon_1(\epsilon_2+\epsilon_3)n_1^2 - \epsilon_2(\epsilon_3+\epsilon_1)n_2^2 - \epsilon_3(\epsilon_1+\epsilon_2)n_3^2 + \epsilon_1\epsilon_2\epsilon_3, \tag{3}$$

where the ϵ_i's (for $i=1$, 2, and 3) are the respective principal dielectric constants and n_i (for $i=1$, 2, or 3) is the ith component of the variable $\mathbf{n}$. It is well known in the literature (see, for example, [2]) that the WS has two sheets, one inner and one outer, touching each other at four symmetrical singular points, with the appearance of "dimples." Furthermore, it is also known [2] that, around each dimple, there is a circle on the surface where the normals are all parallel to one another, the direction of which defines the corresponding optical ray axis in the literature. In other words, all the points on the circle share a common tangent plane. This is exactly the third case we have just described in the last section. As a result, the radiation intensity along any one of the two optical ray axes should be $1/r$-dependent. For illustration, the section of the WS in one of the principal coordinate planes, the X_1X_3 plane, is shown in Fig. 1 by the solid curves (the circle and the ellipse) while a three-dimensional perspective plot of the WS is drawn in Fig. 2, purposely with a window to see the inner sheet, where the axes have been chosen such that $\epsilon_1 < \epsilon_2 < \epsilon_3$. Note that the Z axis gives the direction of the optical ray axis in the X_1X_3 plane, N is the singular point in the first quadrant, and the circle around it, indicated by the two dots in Fig. 1, is shown in Fig. 2; for simplicity, only one optical ray axis is shown. We have set $\epsilon_1=1.5$, $\epsilon_2=3$, and $\epsilon_3=6$ in the figures to exaggerate the dimple and the circle on the WS. Values of ϵ_i's for many biaxial crystals [9] are in fact quite close to one another. A few of them are given in

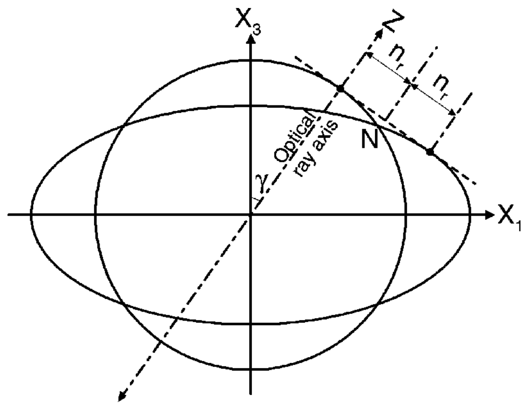

FIG. 1. Section of the WS in the principal X_1X_3 coordinate plane, where γ is the angle between the optical ray axis (indicated by the Z axis) and the X_3 axis, n_r is the radius of the circle, and the dashed line through the two dots shows the common tangent plane.

Table I, where we have also given the radius of the circle and the angle of the optical ray axis with respect to the X_3 axis according to

$$n_r = \frac{1}{2}\sqrt{(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1)/\epsilon_2} \tag{4}$$

and

$$\gamma = \tan^{-1}\sqrt{(\epsilon_2 - \epsilon_1)/(\epsilon_3 - \epsilon_2)}, \tag{5}$$

respectively. (See also Figs. 1 and 2.) The radius can serve as a measure of the closeness of the principal indices of refraction to one another.

Mathematical derivations of the far field with the polarization properties and the radiation intensity on and around an optical ray axis in a biaxial crystal have been obtained and will be presented elsewhere [10]. Yet, making use of the idea of effective area and the almost equality of the three ϵ_i's, we can easily estimate the unusual radiation intensity I_z relative to the usual one (I_{iso}, say) in an isotropic medium of average dielectric constant $\bar{\epsilon}$ defined by $(\epsilon_1 + \epsilon_2 + \epsilon_3)/3$. In the

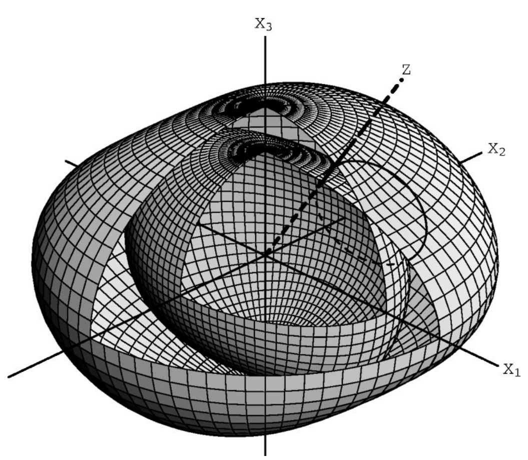

FIG. 2. A three-dimensional perspective plot of the WS with the circle and the optical ray axis (indicated by the Z axis).

TABLE I. Some biaxial crystals with their principal indices of refraction [9] (at sodium D line, 589.2 nm) and other relevant parameters.

Crystal	$\sqrt{\epsilon_1}$	$\sqrt{\epsilon_2}$	$\sqrt{\epsilon_3}$	γ	n_r (units of 10^{-2})	η
Gypsum	1.520	1.523	1.530	33.17°	0.4585	7346
Vivianite	1.579	1.603	1.633	41.11°	2.686	225.4
Sulfur	1.950	2.043	2.240	33.56°	13.70	11.23
Aragonite	1.531	1.682	1.686	80.54°	2.403	286.8
Malachite	1.655	1.875	1.909	67.85°	8.429	25.90
Stibnite	3.194	4.303	4.460	67.86°	39.30	2.641

isotropic case, the WS is spherical so each principal curvature (i.e., κ_u) equals $\omega/(c\sqrt{\bar{\epsilon}})$ and the effective area is $2\pi c\sqrt{\bar{\epsilon}}/(\omega z)$; we have to double the result because the WS is degenerate. On the other hand, the effective area for the unusual case is equal to the circumference of the circle multiplied by the effective width transverse to the circle; it is $2\pi n_r\sqrt{2\pi c\sqrt{\bar{\epsilon}}/(\omega z)}$, where the curvature transverse to the circle has been taken to be the same as that in the isotropic case because of the closeness of the three ϵ_i's. Squaring the ratio of the two effective areas, we therefore have

$$\frac{I_z}{I_{iso}} = \frac{z}{\eta\lambda_0}, \tag{6}$$

where $\eta \equiv \sqrt{\bar{\epsilon}}/\pi^2 n_r^2$ and λ_0 is the wavelength in vacuum. Note that η, also given in Table I, tells the distance (in units of λ_0) from the source beyond which the unusual distance-dependent radiation is increasingly larger than the usual radiation expected in an isotropic medium. In passing, we should also point out that, to conserve energy, the radiation flux lines near the optical ray axis are no longer straight lines [10].

IV. A PROPOSED EXPERIMENT AND CONCLUSION

We now propose an experimental detection with the setup schematically shown in Fig. 3. A triangular plate of biaxial crystal, say of size about 10 cm, is used. It has been cut so that one of its faces (BC in the figure) is perpendicular to an

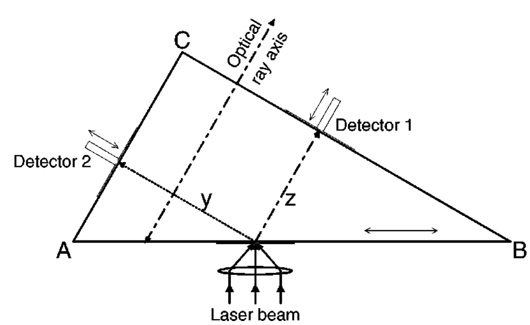

FIG. 3. A sketch of the experimental setup for the detection of the unusual $1/r$-dependent radiation intensity along one of the optical axes in a biaxial crystal.

optical ray axis. A steady-state laser beam is then highly focused at a point on the bottom face (AB in the figure) while a detector (detector 1 in the figure) to collect the exit light is placed close to the BC face at a point along the optical ray axis from the focus on the AB face. The distance between the focus and the exit point can be varied by allowing the plate to move along the direction of AB while at the same time moving the detection system along the optical ray axis accordingly.

As well known from Huygens' principle, the focused spot at the bottom face can be treated as a source. Hence, the detected intensity at the exit point, according to the prediction of Lighthill's theory, should be inversely proportional to distance, i.e., it has the unusual $1/z$ dependence instead of the usual $1/z^2$ dependence, where z is the distance between the focus and the exit point along the optical ray axis as shown in the figure. For the effect to be pronounced, the focus should be as small as possible, say 1 μm in size and, furthermore, there should be a diaphragm with a small hole on the BC-face in front of the detector to guarantee that the light collected is from the laser focus along the optical ray axis. Note that the distance z is of the order of 10 cm, which is much larger than the Rayleigh range [11].

For comparison, one can place another detector (detector 2 in the figure) at an exit point on the third face (the AC face in the figure) along a direction (the y direction, say) other than the optical ray axis in a similar fashion. The intensity should have the usual $1/y^2$ dependence.

We have explained why the unusual $1/r$-dependent radiation intensity exists along the optical ray axis in a biaxial crystal and we have proposed an experiment to detect it. Admittedly, the ''source'' in the experiment is not a real current source, but it serves the purpose. A similar result can be obtained by treating it as a diffraction problem with the use of a dyadic Green's function particularly along the direction of the optical ray axis. Finally, it should be noted that we have avoided in this article the mathematical calculations which, together with the far field results around and away from the axis and the polarization effects, will be presented elsewhere.

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